Midterm - Probability II (2025-26)
Time: 2 hours.
Attempt all questions. The total marks is 30.

1. Consider a simple symmetric random walk S_n on $\mathbf{Z}$ with probability of jumping to the right and left equal to $\frac{1}{2}$. Starting at $S_0 = 0$ what is the probability that the walk does not hit 3 up to time 100 and $S_{100} = 50$? **[5 marks]**

2. Consider a simple symmetric random walk S_n on $\mathbf{Z}$ with probability of jumping to the right and left equal to $\frac{1}{2}$. Let M_{2N} be the time index at which $S_0, S_1, \cdots, S_{2N}$ achieves its maximum value for the *first* time, that is $M_{2N} = i$ if $S_1 < S_i$, $S_2 < S_i, \cdots, S_{i-1} < S_i$, $S_{i+1} \le S_i$, $S_{i+2} \le S_i, \cdots, S_{2N} \le S_i$. Show that

$$P\left(M_{2N} = 2k\right) = \frac{1}{2} P\left(S_{2k=0}\right) \cdot P\left(S_{2(N-k)} = 0\right). \qquad \textbf{[7 marks]}$$

   HINT: You may use the basic lemma without proving it:

$$P\left(S_1 \neq 0,\, S_2 \neq 0, \cdots, S_{2k} \neq 0\right) = P\left(S_{2k} = 0\right).$$

3. Fix $0 < p < 1$ and consider independent $X_i \sim$ Bernoulli (p) random variables, i.e. $P(X_1 = 1) = p$ and $P(X_1 = 0) = q = 1 - p$.

$$a_n = P\left(\sum_{i=1}^{n} X_i \text{ is even}\right),$$

   with $a_0 = 1$.

   (a) Show $a_n = qa_{n-1} + p(1 - a_{n-1})$. **[3 marks]**

   (b) Use the above to show that $A(s) := \sum_{n=0}^{\infty} a_n s^n$ is given by

$$A(s) = \frac{1}{2}\left[(1-s)^{-1} + \{1 - (q-p)s\}^{-1}\right]. \qquad \textbf{[3 marks]}$$

   (c) Show $a_n = \frac{1}{2}\left[1 + (q-p)^n\right]$. **[3 marks]**

4. Consider a simple symmetric random walk S_n on $\mathbf{Z}$ with probability of jumping to the right and left equal to $\frac{1}{2}$. Suppose the walk starts at 50. What is the expected time for the walk to hit the set $\{0, 100\}$? **[9 marks]**
   HINT: Find a recursive relation for D_z: the expected time of the walk to hit $\{0, 100\}$ starting at z.